

# Stable charged cosmic strings

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We study the quantum stabilization of a cosmic string by a heavy fermion doublet in a reduced version of the standard model. We show that charged strings, obtained by populating fermionic bound state levels, become stable if the electro-weak bosons are coupled to a fermion that is less than twice as heavy as the top quark. This result suggests that extraordinarily large fermion masses or unrealistic couplings are not required to bind a cosmic string in the standard model. Numerically we find the most favorable string profile to be a simple “trough” in the Higgs vev of radius  $\approx 10^{-18}$  m. The vacuum remains stable in our model, because neutral strings are not energetically favored.

*Introduction* Various field theories suggest the existence of string-like configurations, which are the particle physics analogues of vortices or magnetic flux tubes in condensed matter physics. They are called cosmic (or Z-)strings to distinguish them from the fundamental variables in string theory and to indicate that they can stretch over cosmic length scales. They can have significant cosmological effects [1] and thus may be relevant to the early universe. Stable strings within the standard model of particle physics would be particularly interesting because they could be observable today.

In the standard model, string configurations [2–4] are not topologically stable and thus can only be stabilized dynamically. Here we focus on the role heavy fermions can play in this stabilization. Since fermions can lower their energy by binding to the string, their binding energy can overcome the classical energy required to form the string background. However, once we include the contribution to the energy from bound fermions, we must also include the contribution from the distortion of the entire fermion spectrum, i.e. the vacuum polarization energy, since both contributions enter at order  $\hbar$ .

A string configuration with a vortex structure introduces non-trivial behavior at spatial infinity. This property invalidates the straightforward application of standard methods to compute the vacuum polarization energy. Recently we have shown how to carry out such a calculation by choosing a particular gauge [5, 6]. We are thus in a position to consistently include fermionic contributions to the dynamical stabilization of cosmic strings.

Naculich [7] has shown that in the limit of weak coupling, fermion fluctuations destabilize the string. The quantum properties of Z-strings have been connected to non-perturbative anomalies [8]. A first attempt at a full calculation of the quantum corrections to the Z-string energy was carried out in ref. [9]. Those authors were only able to compare the energies of two string configurations, rather than comparing a single string configuration to the vacuum; these limitations arise from the non-trivial behavior at spatial infinity. The fermionic vacuum polarization energy of the Abelian Nielson–Olesen

vortex has been estimated in ref. [10] with regularization limited to the subtraction of the divergences in the heat-kernel expansion. Quantum energies of bosonic fluctuations in string backgrounds were calculated in ref. [11]. Previously, we have pursued the idea of stabilizing cosmic strings by populating fermionic bound states in a  $2 + 1$  dimensional model [12]. Many such bound states emerge and some configurations even induce an exact zero-mode [7]. Nonetheless, stable configurations were only obtained for extreme values of the model parameters. In  $3 + 1$  dimensions, stability is more likely because quantization of the momentum parallel to the symmetry axis yields an additional multiplicity of bound states.

*Model and Ansatz* We consider a model of the electroweak interactions in  $D = 3 + 1$  dimensions with some technical simplifications, which we will justify *a posteriori*. We set the Weinberg angle to zero, so that electromagnetism is decoupled from the theory and the  $SU(2)$  gauge bosons are degenerate. We also neglect QCD interactions, though we include the  $N_C = 3$  color degeneracy in computing the fermion contribution to the string energy. Finally, we consider a single heavy doublet that is degenerate in mass, neglecting CKM mixing and mass splitting within the doublet. The classical Higgs and gauge fields are described by the Lagrangian

$$\mathcal{L}_{\phi,W} = -\frac{1}{2}\text{tr}(G^{\mu\nu}G_{\mu\nu}) + \frac{1}{2}\text{tr}(D^\mu\Phi)^\dagger D_\mu\Phi - \frac{\lambda}{2}\text{tr}(\Phi^\dagger\Phi - v^2)^2, \quad (1)$$

where  $\Phi$  represents the Higgs doublet  $\phi = (\phi_+, \phi_-)$  as a matrix,  $\Phi = \begin{pmatrix} \phi_0^* & \phi_+ \\ -\phi_+^* & \phi_0 \end{pmatrix}$ , the gauge coupling constant enters via the covariant derivative  $D^\mu = \partial^\mu - igW^\mu$ , and the  $SU(2)$  field strength tensor is  $G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig[W_\mu, W_\nu]$ . We then have the fermion Lagrangian

$$\mathcal{L}_\Psi = i\bar{\Psi}(P_L\not{D} + P_R\not{\partial})\Psi - f\bar{\Psi}(\Phi P_R + \Phi^\dagger P_L)\Psi, \quad (2)$$

where the Yukawa coupling  $f$  controls the strength of the Higgs–fermion interaction, which generates the fermion mass. Our model is thus characterized by the fermion

mass  $m_f = fv$ , the gauge boson mass  $m_W = gv/\sqrt{2}$ , the Higgs mass  $m_H = 2v\sqrt{\lambda}$ , and the Higgs vacuum expectation value (vev)  $v$ . When we introduce the fermionic quantum corrections, we impose on-shell renormalization conditions, in which we hold fixed  $m_H$ ,  $v$ , and the residue of the pole in each particle's propagator. These choices exhaust the available counterterms, so we have to adjust the gauge coupling  $g$  to match the physical gauge boson mass. Since we neglect boson loops, this renormalization scheme also leaves the fermion mass unchanged.

We construct the string as a classical background field that is translationally invariant in the  $z$ -direction. We work in Weyl gauge  $W^0 = 0$  and also introduce a parameter  $\xi_1$  that allows us to include a gauge field with winding number  $n$ . We set  $n$  to unity in the actual calculations. The gauge and Higgs fields are then

$$\vec{W} = n s \frac{f_G(\rho)}{g\rho} \hat{\varphi} \begin{pmatrix} s & ic e^{-in\varphi} \\ -ic e^{in\varphi} & -s \end{pmatrix}$$

and  $\Phi = v f_H(\rho) \begin{pmatrix} s e^{-in\varphi} & -ic \\ -ic & s e^{in\varphi} \end{pmatrix}, \quad (3)$

where  $s = \sin(\xi_1)$  and  $c = \cos(\xi_1)$ , and  $(\rho, \varphi)$  are polar coordinates in the plane perpendicular to the string axis. This ansatz yields the classical energy per unit length

$$E_{cl} = 2\pi \int_0^\infty d\rho \rho \left[ \frac{2}{g^2} \left( \frac{f'_G}{\rho} \right)^2 + v^2 (f'_H)^2 + \frac{v^2}{\rho^2} f_H^2 (1 - f_G)^2 + \lambda v^4 (1 - f_H^2)^2 \right], \quad (4)$$

where primes denote derivatives with respect to  $\rho$ . Variational width parameters  $w_H$  and  $w_G$  enter through the respective profile functions for each field,

$$f_H(\rho) = 1 - \exp(-\rho/w_H), \quad f_G(\rho) = 1 - \exp(-\rho^2/w_G^2). \quad (5)$$

*Energy Considerations* We compute the total binding energy per unit length as a sum of three terms:

$$E_{tot} = E_{cl} + N_C(E_{vac} + E_b). \quad (6)$$

The classical energy per unit length depends on the model parameters and the variational parameters  $w_H$ ,  $w_G$  and  $\xi_1$ . The two contributions in eq. (6) proportional to  $N_C$  summarize the fermionic effects. We measure all dimensionful quantities in comparison to appropriate powers of  $m_f$ , so that  $E_{vac}$  and  $E_b$  only depend on the ansatz parameters  $w_H$ ,  $w_G$  and  $\xi_1$ .<sup>1</sup> QCD effects only enter via the degeneracy factor  $N_C$ : since the considered energy scales are well above the QCD scale, these interactions can be neglected due to asymptotic freedom.

The fermionic effects are computed from the single particle Dirac Hamiltonian in the two-dimensional subspace orthogonal to the symmetry axis of the string<sup>2</sup>. The profiles  $f_G$  and  $f_H$  act as potentials in this Hamiltonian.

The vacuum polarization energy per unit length in the string background  $E_{vac}$  is the computationally most expensive part of the calculation. It is computed from the scattering solutions to the single particle Hamiltonian using the spectral method [5, 13, 14]. This part involves a number of technical subtleties associated with the long-ranged string potential [6, 15].

Finally, the single particle Hamiltonian has many bound state solutions; for  $\xi_1 = \frac{\pi}{2}$  there exists an exact zero mode. By explicitly populating these bound states, we add charge to the string. Numerically, we compute bound state energies in the string background by discretizing the reduced two-dimensional system in a finite box and diagonalizing the Hamiltonian matrix numerically. Let  $\epsilon_i \leq m_f$  be an eigenvalue of the two-dimensional Dirac Hamiltonian. Then a state has energy  $[\epsilon_i^2 + p^2]^{1/2}$ , where  $p$  is its conserved momentum along the symmetry axis. To count the populated states, we introduce a chemical potential  $\mu$  such that  $\min\{|\epsilon_i|\} \leq \mu \leq m_f$ . States with  $[\epsilon_i^2 + p^2]^{1/2} < \mu$  are filled while states with  $[\epsilon_i^2 + p^2]^{1/2} > \mu$  remain empty, which gives a Fermi momentum  $P_i(\mu) = [\mu^2 - \epsilon_i^2]^{1/2}$  for each bound state. According to the Pauli exclusion principle we can occupy each state only once. This yields the charge density per unit length of the string

$$Q(\mu) = \frac{1}{\pi} \sum_{\epsilon_i \leq \mu} P_i(\mu), \quad (7)$$

where the sum runs over all bound states available for a given chemical potential.<sup>3</sup>

Eq. (7) can be inverted to give  $\mu = \mu(Q)$ . In numerical computations we prescribe the left-hand-side of eq. (7) and increase  $\mu$  from  $\min\{|\epsilon_i|\}$  until the right-hand-side matches. From this value  $\mu = \mu(Q)$ , the binding energy per unit length

$$E_b(Q) = \frac{1}{\pi} \sum_{\epsilon_i \leq \mu} \int_0^{P_i(\mu)} dp \left[ \sqrt{\epsilon_i^2 + p^2} - m_f \right] \quad (8)$$

can be computed as a function of the prescribed charge. In this manner the total energy becomes a function of the charge density of the string. Filling the available states up to a common chemical potential minimizes  $E_b$ : if the

<sup>1</sup> There is a weak (logarithmic) dependence on the model parameters introduced via the on-shell renormalization conditions; it is small for the values of the coupling constants we consider.

<sup>2</sup> We refrain from displaying this Hamiltonian, which we extract from eq. (2). For actual computations a specific gauge must be adopted, complicating its presentation [6, 15].

<sup>3</sup> Ambiguities in this relation due to different boundary conditions at the end of the string show up at subleading order in  $1/L$  where  $L$  is the length of the string and can thus be safely ignored.

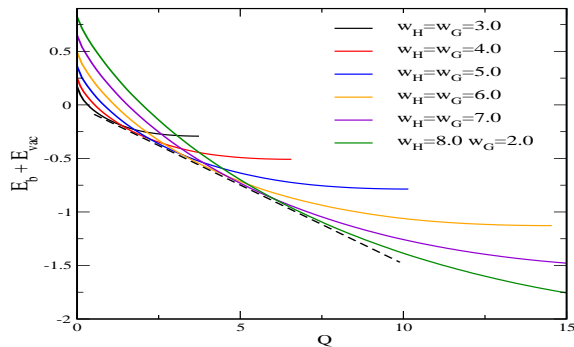


FIG. 1: (Color online) Total bound state and vacuum energy per unit length as a function of charge density per unit length, in units of the fermion mass, for  $\xi_1 = 0.4\pi$ . The dotted line indicates the minimal fermionic contribution to the energy.

towers of states built upon two different  $\epsilon_i$  had different upper limits, the energy would be lowered by moving a state from the tower with the larger limit to that with the lower one, without changing the charge.

Our central task is to find Higgs–gauge field configurations that yield  $E_{\text{tot}} < 0$  for a prescribed value of the charge density,  $Q$ . In doing so, we must take care that any binding we observe is not an artifact of the Landau pole, which eventually sends  $E_{\text{vac}}$  to minus infinity as  $w_H$  and/or  $w_G$  tend to zero. It arises because in our approximation (neglecting contributions from bosonic loops) the model is not asymptotically free. Once we identify a configuration and parameter set with interesting numerical results we use a method similar to that of ref. [16] to ensure that the Landau pole contribution is negligible.

**Results** The similarity to the standard model suggests the model parameters

$$g = 0.72, v = 177 \text{ GeV}, m_H = 140 \text{ GeV}, f = 0.99. \quad (9)$$

The Yukawa coupling estimate is obtained from the top–quark mass  $m_t = 175 \text{ GeV}$ . To consider a fourth generation with a heavy fermion doublet that couples to the standard model bosons, we will vary the Yukawa coupling but keep all other model parameters fixed.

For the configurations we consider, the classical energy, eq. (4), is dominated by the Higgs potential contribution, which scales as  $\lambda w_H^2 / (f^4 N_C)$  compared to the fermionic contributions. As  $\xi_1 \rightarrow 0$ , the gauge field contribution goes to zero, so this choice is favored classically. We will see that  $\xi_1 \approx 0$  remains favored when  $E_{\text{vac}}$  and  $E_b$  are included, so that the stable charged string obtained in our model is simply a “trough” in the Higgs vev, without significant gauge field contributions.

We give all numerical results in units of  $m_f$  or  $1/m_f$  as appropriate. In fig. 1 we display the fermion contributions for various sets of ansatz parameters. These lines terminate at an end point where all available bound states (for all longitudinal momenta) are populated and the charge cannot be increased any further. The fermion

contributions favor a wide string for large charges, while they cause the string to shrink for small charges. For very small charges, corresponding to small widths, the calculation is unreliable because of the Landau pole.<sup>4</sup>

When we add more configurations, we observe a linear relation between the charge and the *minimal* fermion contribution to the energy, even though for any given configuration, the fermion energy depends quadratically on the charge, *cf.* eqs. (7) and (8). This linear dependence arises from a delicate balance between the vacuum polarization (which determines the  $y$ –intercept for a given configuration) and the binding energies (which determine the  $Q$  dependence). Figure 1 also suggests that the width of the Higgs profile,  $w_H$ , is the dominating scale (which is corroborated in fig. 3, where  $E_b + E_{\text{vac}}$  is seen to be nearly independent of  $\xi_1$ , and thus of  $w_G$ .) Both the number of two–dimensional bound states and the magnitude of their binding energies  $\epsilon_i - m_f$  vary roughly linearly with  $w_H$ . As a result, the minimal fermion contribution scales quadratically with  $w_H$ , as the classical energy does. To decide if the string is stable we have to compare the leading scaling with  $w_H^2$  in  $E_{\text{cl}}$  and  $E_{\text{vac}} + E_b$ . For large widths, the string is stable if the resulting coefficient of the scaling with  $w_H^2$  is negative. For physically motivated parameters, eq. (9), the classical energy dominates and there is no binding for any charge. However, as mentioned above, the relative contribution from  $E_{\text{cl}}$  decreases like  $1/f^4$ . So even a moderate increase of the fermion mass could lead to binding. We remark that extrapolating the straight line in fig. 1 predicts that the vacuum energy should vanish for very narrow strings, as we would expect. This estimate overcomes the Landau pole obstacles that arise in a direct computation.

To search for a stable string of fixed charge  $Q$ , we have computed the vacuum polarization energy and the bound state energies from the two–dimensional Hamiltonian for several hundred configurations characterized by specific values of the ansatz parameters  $w_H$ ,  $w_G$  and  $\xi_1$ . We then prescribe the charge  $Q$  and, for those configurations that can accommodate it according to eq. (7), we compute the binding energy as in eq. (8). Once we have computed the fermionic contribution to  $E_{\text{tot}}$ , the classical energy is a simple spatial integral, which requires a negligible amount of additional computation. As a result, in this procedure it is most efficient to vary the Yukawa coupling. For a given charge, we then have a large set of configurations that are labeled by given (discrete) values of the variational parameters. We scan this set for the minimal total energy. If the variational parameters covered the full configuration space, this treatment would be

<sup>4</sup> The problem arises for widths much less than unity and coupling constants of order five or larger. In our numerical search for stable configurations we only consider  $w_H \geq 2$  and  $w_G \geq 2$ .

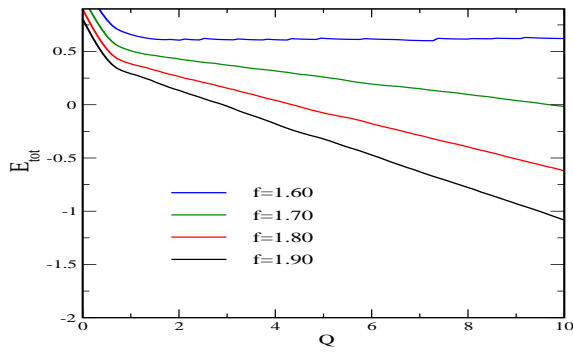


FIG. 2: (Color online) Total energy per unit length of optimal string configurations as a function of charge per unit length, in units of the fermion mass.

equivalent to the self-consistent construction of the minimal energy configuration. With our restriction to the variational space, however, we only find an upper limit to the exact minimum; if our treatment detects a bound configuration, the existence of a stable charged cosmic string is established.

In fig. 2, we show the full energy per unit length  $E_{\text{tot}}$  as a function of the charge density per unit length for a variety of Yukawa couplings  $f$ . The sharp increase at small  $Q$  is an artifact of the restriction of the ansatz parameters, *cf.* footnote 4. Increasing the Yukawa coupling from its top-quark value decreases the relative contribution from  $E_{\text{cl}}$  to  $E_{\text{tot}}$ . We see that at  $f \approx 1.6$  the large width pieces from  $E_{\text{cl}}$  and  $E_{\text{vac}} + E_{\text{b}}$  approximately cancel. Increasing the Yukawa coupling only slightly more, *e.g.* to  $f \gtrsim 1.7$ , yields a negative total energy per unit length at large charge densities, which indicates that the string is lighter than the corresponding density of free fermions. This limit corresponds to a fermion mass of about 300 GeV with a typical width for the stable charged string of about  $10^{-18}\text{m}$  ( $w_H \approx 4/m_f$ ).

Surprisingly, we find that the fermion contribution to the energy is nearly independent of the ansatz parameter  $\xi_1$ , as shown in fig. 3. Even though the bound state spectrum varies strongly with  $\xi_1$ , and  $E_{\text{vac}} + E_{\text{b}}$  depends only weakly on  $\xi_1$  [6], there are subtle cancellations within the bound state spectrum itself that yield such a tiny gauge field dependence of the fermion energy. As  $g \ll f$ , the gauge field terms increase  $E_{\text{cl}}$  for  $\xi_1 \neq 0$ . Hence  $E_{\text{tot}}$  is minimized for  $\xi_1 \approx 0$  in the cases we have studied.

*Discussion* We have seen that a heavy fermion doublet can stabilize a nontrivial string background in a simplified version of the electroweak standard model for a non-zero fixed charge density. Light fermions would contribute only weakly to the binding of the string, since their Yukawa couplings are small. As a result, we can add them to our model, *e.g.* to accommodate anomaly cancellation, without significantly changing the result. The resulting configuration is essentially of pure Higgs structure. Any additional (variational) degree of freedom can

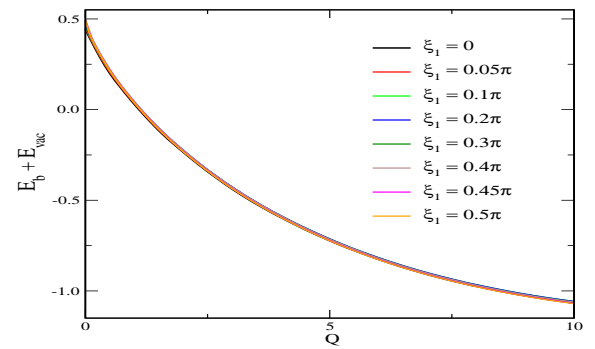


FIG. 3: (Color online) Fermionic contribution to the string binding energy per unit length as a function of charge density per unit length, in units of the fermion mass, for a variety of values of  $\xi_1$  and  $w_H = 6.0$  and  $w_G = 6.0$ .

only lower the total energy. Hence embedding this configuration in the full standard model, with the  $U(1)$  gauge field included, also yields a bound object. We see binding set in at  $m_f \approx 300\text{ GeV}$ , which is still within the range of energy scales at which the standard model should provide an effective description of the relevant physics, and also within the range to be probed at the LHC. For such fermion masses, recent calculations have also suggested the potential stability of multi-fermion bound states in a Higgs background [17, 18].

The fermion bound states carry non-zero angular momenta, implying that the bound state wave-functions depend on the azimuthal angle. This might induce a more complicated spatial structure of the string configuration than the one adopted in eq. (3). In particular, the cylindrical analog of spherical “hedgehog” configurations, representing a Higgs field with unit winding within a  $U(1)$  subgroup of the full  $SU(2)$  isospin group, could be an interesting extension of our work. Such alterations can only lower the total energy, however.

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